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The L^2 geometry of spaces of harmonic maps $S^2 \to S^2$ and $\mathbb{R}P^2 \to \mathbb{R}P^2$

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Abstract

Harmonic maps from S^2 to S^2 are all weakly conformal, and so are represented by rational maps. This paper presents a study of the L^2 metric γ on M_n, the space of degree n harmonic maps $S^2 \to S^2$, or equivalently, the space of rational maps of degree n. It is proved that γ is Kähler with respect to a certain natural complex structure on M_n. The case $n = 1$ is considered in detail: explicit formulae for γ and its holomorphic sectional, Ricci and scalar curvatures are obtained, it is shown that the space has finite volume and diameter and codimension 2 boundary at infinity, and a certain class of Hamiltonian flows on M_1 is analysed. It is proved that $\tilde{\mathrm{M}}_n$, the space of absolute degree n (an odd positive integer) harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$, is a totally geodesic Lagrangian submanifold of M_n, and that for all $n \geq 3$, $\tilde{\mathrm{M}}_n$ is geodesically incomplete. Possible generalizations and the relevance of these results to theoretical physics are briefly discussed.

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1. Introduction

In theoretical physics, one often regards harmonic maps $(M, g) \to (N, h)$, from a Riemannian manifold of dimension 2, as static solutions of the so-called nonlinear σ-model on space–time $(M \times \mathbb{R}, \eta)$, where $\eta = \mathrm{d}t^2 - g$ is the Lorentzian pseudometric. Those harmonic maps which minimize energy within their homotopy class are usually called “lumps” in this context, because generically their energy density is localized in lump-like structures distributed over M. In many cases of interest, the homotopy classes of maps $\phi : M \to N$ are

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labelled by the topological degree of ϕ, and the moduli space of static degree n lumps, M_n, is a smooth, finite-dimensional manifold. There is a natural Riemannian metric on M_n, namely the L^2 metric, which assigns to each pair of tangent vectors $X, Y \in T_\phi \mathsf{M}_n \subset \Gamma(\phi^* TN)$ the inner product

$$\gamma(X, Y) = \int_M d\mu_g \, h_\phi(X, Y), \tag{1.1}$$

where $d\mu_g$ denotes the area measure on (M, g). The physical interpretation of this metric is that it is the restriction to M_n of the symmetric bilinear form defined by the kinetic energy functional of the parent σ-model. Note that, unlike the harmonic map energy, the kinetic energy (and hence γ) depends on g, not just the conformal class of g.

This paper presents a study of this metric in the cases $M = N = S^2$ and $M = N = \mathbb{R}P^2$ with their canonical metrics. These cases are convenient because one has complete, explicit parameterizations of the harmonic maps in terms of rational functions. We will focus particularly on the simplest nontrivial case, degree 1 maps $S^2 \to S^2$, obtaining a quite thorough understanding of its L^2 geometry. The choice $N = S^2$ or $\mathbb{R}P^2$ is rather natural from the stand-point of physics since the order parameters of ferromagnets and nematic liquid crystals are S^2- and $\mathbb{R}P^2$-valued, respectively [36]. Previously, the algebraic topology of spaces of rational maps has been studied by Segal [31] and Guest et al. [9], and the algebraic topology of spaces of harmonic maps $S^2 \to S^m$ and $S^2 \to \mathbb{R}P^m$ by Furuta et al. [7]. The differential topology of spaces of harmonic maps $S^2 \to S^{2m}$ and $S^2 \to \mathbb{C}P^m$ has been studied by Bolton and Woodward [3] and Lemaire and Wood [19], respectively. The present paper may be considered complementary to this body of work.

The physical motivation behind this study is that σ-model lumps are in many ways analogous to topological solitons in relativistic gauge theories, such as BPS monopoles and Abelian Higgs vortices. In the S^2 case, for example, lumps attain a Bogomol'nyi type topological lower bound on energy within their homotopy class, and consequently satisfy a first order "self-duality" equation (namely, the Cauchy–Riemann equation). Manton conjectured [22] that the slow motion of n BPS monopoles is well approximated by geodesic flow with respect to the L^2 metric on the n-monopole moduli space. This conjecture was extended to lumps by Ward [37], and has since been formulated and proved rigorously for monopoles and vortices by Stuart [34,35]. The metric in the case $n = 2$, $M = \mathbb{R}^2$, $N = S^2$ was investigated numerically by Leese [18]. So the physical motivation behind the present work is the hope that the L^2 metrics will shed light on slow lump dynamics in the parent σ-model, as the Atiyah and Hitchin [1] and Samols [30] metrics have done for monopole and vortex dynamics. Of course, they remain interesting and natural geometric structures in their own right.

The rest of the paper is structured as follows. Let M_n, $n \in \mathbb{Z}$, denote the space of degree n harmonic maps $S^2 \to S^2$. In Section 2, we give a simple, concrete proof that (M_n, γ) is Kähler with respect to the complex structure induced by a natural open inclusion $\mathsf{M}_n \subset \mathbb{C}P^{2n+1}$. This result was previously conjectured (in a rather more general setting) by Ruback [28], who gave a very persuasive formal argument in its favour. In Section 3 we show that the Kähler property, along with the isometry group, almost completely determines the L^2 metric on M_1. Specifically, we show that any Kähler metric on M_1 invariant under the isometry group of γ is determined by a single function of one variable, rather than 21 functions of six variables, as for a generic metric in six dimensions. An explicit formula for γ is given, and it is shown that M_1, although noncompact, has finite volume and diameter. It is shown also that the boundary of (M_1, γ) at infinity has codimension 2.

In Section 4, the curvature properties of M_1 are studied. Explicit formulae for the holomorphic sectional curvatures of a certain unitary frame and for the Ricci and scalar curvatures are derived. It is shown that the holomorphic sectional and scalar curvatures are unbounded above, and conjectured that the Ricci curvature is positive definite. The relevance of these results to quantum lump dynamics is discussed.

It is natural to regard the Kähler form Ω on M_n as a symplectic form and study the symplectic geometry of (M_n, Ω). Such symplectic geometry has recently been used to study vortex dynamics in a nonrelativistic version of the Abelian Higgs model, for example [27]. In Section 5, the most general physically meaningful Hamiltonian flow on (M_1, Ω) is analysed, and the corresponding one lump dynamics described.

In Section 6, we address the geometry of spaces of harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$. Eells and Lemaire [6] have shown that, if nonconstant, such maps are classified homotopically by a certain odd positive integer, which we shall call the absolute degree (see Section 6 for a definition). In Section 6 it is proved that $\tilde{\mathrm{M}}_n$, the space of absolute degree n harmonic maps, is naturally identified with a certain totally geodesic Lagrangian submanifold of M_n, where the symplectic form is again taken to be the Kähler form. Further, it is shown that for all $n \geq 3$, $\tilde{\mathrm{M}}_n$ is geodesically incomplete, while $\tilde{\mathrm{M}}_1$ is compact.

Finally, in Section 7 we speculate on possible generalizations of this work. As an example, it is shown that the L^2 metric on the space of degree 2 elliptic functions is naturally Kähler.

2. The Kähler property of M_n

By the Hopf degree theorem [10], homotopy classes of continuous maps $\phi : S^2 \to S^2$ are labelled by their topological degree $n \in \mathbb{Z}$. A well-known argument of Lichnerowicz [20] (rediscovered independently by physicists Belavin and Polyakov [2] and Woo [40]) shows that in the degree n class the harmonic map energy satisfies $E[\phi] \geq 2\pi|n|$, with equality if and only if ϕ is holomorphic ($n \geq 0$) or antiholomorphic ($n < 0$). Since harmonic maps are by definition local extremals of E, (anti)holomorphic maps are harmonic, and furthermore, minimize energy within their class. In fact, all harmonic maps $S^2 \to S^2$ are (anti)holomorphic [41]. Since degree n and $-n$ maps are trivially related by a change of orientation (on domain or codomain), we may, and henceforth will, assume $n \geq 0$ without loss of generality.

Introducing complex stereographic coordinates z, W on domain and codomain, the general degree n harmonic map is

$$W(z) = \frac{a_1 + a_2 z + \cdots + a_{n+1} z^n}{a_{n+2} + a_{n+3} z + \cdots + a_{2n+2} z^n}, \tag{2.1}$$

where $a_i \in \mathbb{C}$ are constants, a_{n+1} and a_{2n+2} both do not vanish, and the numerator and denominator share no common roots. So M_n is the space of degree n rational maps. Clearly, any point $(\xi a_1, \ldots, \xi a_{2n+2}) \in \mathbb{C}^{2n+2}$, $\xi \in \mathbb{C}^\times := \mathbb{C}\backslash\{0\}$, determines the same rational map as $(a_1, \ldots, a_{2n+2})$, so one may identify each rational map with a point in $\mathbb{C}P^{2n+1}$. This gives a natural open inclusion $\mathrm{M}_n \subset \mathbb{C}P^{2n+1}$ (not an identification, since the "no common roots" condition removes a complex codimension 1 algebraic variety from $\mathbb{C}P^{2n+1}$) which we use to equip M_n with a topology and complex structure. This topology is natural in that it coincides with the relative topology of M_n in $C^0(S^2, S^2)$. The metric of interest does *not* derive from the inclusion $\mathrm{M}_n \subset \mathbb{C}P^{2n+1}$, of course, but rather from definition (1.1). We now establish the following theorem.

Theorem 2.1. *For all $n > 0$, (M_n, γ) is Kähler with respect to the complex structure induced by the open inclusion $\mathrm{M}_n \subset \mathbb{C}P^{2n+1}$.*

Proof. On the open set where $a_{2n+2} \neq 0$, we may introduce complex local coordinates $b^\alpha = a_\alpha / a_{2n+2}$, $\alpha = 1, 2, \dots, 2n+1$. We may always arrange that $a_{2n+2} \neq 0$ by a rotation of the codomain, so it suffices to show that γ is Kähler in this coordinate system. Explicitly,

$$\gamma = \gamma_{\alpha\beta}\, \mathrm{d}b^\alpha\, \mathrm{d}\bar{b}^\beta, \tag{2.2}$$

where repeated indices are summed over, and

$$\gamma_{\alpha\beta} = \int_{\mathbb{C}} \frac{\mathrm{d}z\, \mathrm{d}\bar{z}}{(1+|z|^2)^2} \frac{1}{(1+|W|^2)^2} \frac{\partial W}{\partial b^\alpha} \left(\frac{\partial \overline{W}}{\partial b^\beta} \right), \tag{2.3}$$

$$W = \frac{b_1 + b_2 z + \cdots + b_{n+1} z^n}{b_{n+2} + b_{n+3} z + \cdots + z^n}. \tag{2.4}$$

Note that γ is manifestly Hermitian, that is, $\gamma_{\beta\alpha} \equiv \bar{\gamma}_{\alpha\beta}$. Hence, we only need to demonstrate that

$$\frac{\partial \gamma_{\alpha\beta}}{\partial b^\delta} \equiv \frac{\partial \gamma_{\delta\beta}}{\partial b^\alpha}, \qquad \frac{\partial \gamma_{\alpha\beta}}{\partial \bar{b}^\delta} \equiv \frac{\partial \gamma_{\alpha\delta}}{\partial \bar{b}^\beta} \tag{2.5}$$

for all α, β, δ [24]. In fact (2.5) follow immediately from (2.3) and (2.4) provided one may interchange the order of partial derivative and integral in $\partial \gamma_{\alpha\beta} / \partial b^\delta$. But this is an immediate consequence of the following lemma, whose proof is presented in Appendix A.

Lemma 2.2. *Let X be a compact Riemannian manifold, $F : X \times (-\epsilon, \epsilon) \to \mathbb{R}$ be smooth and $f : (-\epsilon, \epsilon) \to \mathbb{R}$ such that*

$$f(x) = \int_X F(\cdot, x).$$

Then

$$f'(0) = \int_X F_2(\cdot, 0),$$

where $F_2 : X \times (-\epsilon, \epsilon) \to \mathbb{R}$ is the partial derivative of F with respect to the second entry.

Applying this to the integrand of (2.3), with x representing the (real or imaginary part of) any of the coordinates b^α, the result is proved. □

Before specializing to the case $n = 1$, we note two facts about M_n. First, (M_n, γ) is geodesically incomplete. This is a special case of a more general result [29]. Second, both domain and codomain spheres are isometric under the group of rotations and reflections of $\mathbb{R}^3$, $\mathrm{O}(3) \cong \mathrm{SO}(3) \cup \overline{\mathrm{SO}(3)}$. Here $\overline{\mathrm{SO}(3)}$ denotes the orientation reversing component. The induced action of $\mathrm{O}(3) \times \mathrm{O}(3)$ on the set of continuous maps $S^2 \to S^2$ decomposes $\mathrm{O}(3) \times \mathrm{O}(3)$ into degree preserving and degree reversing components:

$$\mathrm{O}(3) \times \mathrm{O}(3) \cong [(\mathrm{SO}(3) \times \mathrm{SO}(3)) \cup (\overline{\mathrm{SO}(3)} \times \overline{\mathrm{SO}(3)})] \cup [(\mathrm{SO}(3) \times \overline{\mathrm{SO}(3)}) \cup (\overline{\mathrm{SO}(3)} \times \mathrm{SO}(3))]. \tag{2.6}$$

The degree preserving subgroup, call it G, acts isometrically on (M_n, γ). It is convenient to define $P : \mathrm{M}_n \to \mathrm{M}_n$ such that $P : W(z) \mapsto \overline{W(\bar{z})}$. Then $G \cong \mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\mathrm{Id}, P\}$. We shall denote the identity component of G by G_0.

3. The metric on M_1

In the case $n = 1$, the isometric action of $G_0 \cong \mathrm{SO}(3) \times \mathrm{SO}(3)$ described above has cohomogeneity 1, that is, generic G_0 orbits have codimension 1. This is most easily seen by identifying M_1 with $\mathrm{PL}(2, \mathbb{C})$. Note that the case $n = 1$ is special in that degree 1 rational maps are closed under composition, so M_1 has a natural Lie group structure, namely that of the Möbius group $\mathrm{PL}(2, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$. Explicitly, one identifies a rational map

$$W : z \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} \tag{3.1}$$

with a projective equivalence class of $\mathrm{GL}(2, \mathbb{C})$ matrices,

$$[M] = \left\{ \xi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \xi \in \mathbb{C}^\times \right\}, \tag{3.2}$$

noting that map composition and matrix multiplication correspond under the identification. Then the $\mathrm{PU}(2) \cong \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3)$ subgroup of $\mathrm{PL}(2, \mathbb{C})$ consists of rotations of S^2, so in matrix language G_0 acts on $\mathrm{PL}(2, \mathbb{C})$ by left and right $\mathrm{PU}(2)$ matrix multiplication.

A particularly convenient moving coframe for $\mathrm{PL}(2, \mathbb{C})$ is defined as follows. Let τ_a, $a = 1, 2, 3$ be the standard Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

Then any $[M] \in \mathrm{PL}(2, \mathbb{C})$ has a unique polar decomposition

$$[M] = [U](\Lambda \mathbb{I}_2 + \boldsymbol{\lambda} \cdot \boldsymbol{\tau}), \tag{3.4}$$

where $[U] = \{\pm U\} \in \mathrm{PU}(2)$, $\boldsymbol{\lambda} \in \mathbb{R}^3$, $\lambda = |\boldsymbol{\lambda}|$, $\Lambda = \sqrt{1 + \lambda^2}$ and $\cdot$ denotes the $\mathbb{R}^3$ scalar product [25]. The moving coframe is $\{\mathrm{d}\lambda_a, \sigma_a : a = 1, 2, 3\}$, where σ_a are the left-invariant 1-forms on $\mathrm{PU}(2)$ associated with the basis $\{\mathrm{i}/2\tau_a : a = 1, 2, 3\}$ for $\mathrm{su}(2) \cong T_{[\mathbb{I}_2]}\mathrm{PU}(2)$. So $\mathrm{M}_1 \cong \mathrm{PU}(2) \times \mathbb{R}^3$ as a manifold (though *not* as a group). Physically, the lump parameterized by $([U], \boldsymbol{\lambda})$ should be thought of as located at $-\hat{\boldsymbol{\lambda}} \in S^2$ (where $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}/\lambda$), with "sharpness" λ and internal orientation $[U]$. The action of $([L], [R]) \in \mathrm{PU}(2) \times \mathrm{PU}(2) \cong G_0$ on M_1 in terms of the polar decomposition is

$$([L], [R]) : ([U], \boldsymbol{\lambda}) \mapsto ([LUR], \mathcal{R}\boldsymbol{\lambda}), \tag{3.5}$$

where $\mathcal{R} \in \mathrm{SO}(3)$ is the rotation corresponding to $[R] \in \mathrm{PU}(2)$ (explicitly, it has matrix components $\mathcal{R}_{\mathrm{ab}} = (1/2)\,\mathrm{tr}(\tau_a R^\dagger \tau_b R)$). From this, one sees that the G_0 action indeed has cohomogeneity 1, the orbits being level sets of λ. The orbit space M_1/G_0 may be identified with the radial curve $\Gamma = \{([\mathbb{I}_2], (0, 0, \lambda)) : \lambda \geq 0\}$ of rational maps $W_\lambda : z \mapsto \mu(\lambda)z$, where $\mu(\lambda) = (\Lambda + \lambda)^2$. There is one exceptional orbit, namely $\lambda = 0$, which has codimension 3.

The main aim of this section is to obtain an explicit formula for γ, by applying the following proposition.

Proposition 3.1. *Let τ be a G invariant symmetric $(0, 2)$ tensor on M_1 which is Hermitian ($\tau(JX, JY) \equiv \tau(X, Y)$), and whose J-associated 2-form $\hat{\tau}$ ($\hat{\tau}(X, Y) := \tau(JX, Y)$) is closed. Then there exists a smooth function $A : [0, \infty) \to \mathbb{R}$ such that*

$$\tau = A_1\, \mathrm{d}\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda} + A_2(\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda})^2 + A_3\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} + A_4(\boldsymbol{\lambda} \cdot \boldsymbol{\sigma})^2 + A_5\boldsymbol{\lambda} \cdot (\boldsymbol{\sigma} \times \mathrm{d}\boldsymbol{\lambda}), \tag{3.6}$$

where

$$A_1 = A(\lambda), \qquad A_2 = \frac{A(\lambda)}{1+\lambda^2} + \frac{A'(\lambda)}{\lambda}, \qquad A_3 = \left(\frac{1+2\lambda^2}{4}\right) A(\lambda),$$
$$A_4 = \left(\frac{1+\lambda^2}{4\lambda}\right) A'(\lambda), \qquad A_5 = A(\lambda), \tag{3.7}$$

A' denotes the derivative of A, $\times$ the $\mathbb{R}^3$ vector product and juxtaposition of covectors denotes symmetrized tensor product.

Proof. We first show that the most general G_0 invariant symmetric $(0, 2)$ tensor on M_1 is

$$\begin{aligned}\tau = {} & A_1\, \mathrm{d}\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda} + A_2(\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda})^2 + A_3\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} + A_4(\boldsymbol{\lambda} \cdot \boldsymbol{\sigma})^2 + A_5\boldsymbol{\lambda} \cdot (\boldsymbol{\sigma} \times \mathrm{d}\boldsymbol{\lambda}) \\ & + A_6\boldsymbol{\sigma} \cdot \mathrm{d}\boldsymbol{\lambda} + A_7(\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda})(\boldsymbol{\lambda} \cdot \boldsymbol{\sigma}),\end{aligned} \tag{3.8}$$

where $A_1, \ldots, A_7$ are functions of λ only.

Thus such a τ is G_0 invariant follows from the pulled back action of G_0 on our moving coframe:

$$([L], [R]) : (\mathrm{d}\boldsymbol{\lambda}, \boldsymbol{\sigma}) \mapsto (\mathcal{R}\, \mathrm{d}\boldsymbol{\lambda}, \mathcal{R}\boldsymbol{\sigma}). \tag{3.9}$$

We may prove that (3.8) is the most general G_0 invariant symmetric (0, 2) tensor possible by means of the representation theory of $\mathrm{SO}(N)$. Any such tensor is uniquely determined by the 1-parameter family of symmetric bilinear forms $\tau_\lambda : V_\lambda \oplus V_\lambda \to \mathbb{R}$, where $V_\lambda = T_{W_\lambda}\mathrm{M}_1$, and each τ_λ must be invariant under the isotropy group $H_\lambda < G_0$ of W_λ.

Explicitly,

$$H_\lambda = \begin{cases} \left\{\left(\left[\exp\left(-\frac{\mathrm{i}}{2}\psi\tau_3\right)\right], \left[\exp\left(\frac{\mathrm{i}}{2}\psi\tau_3\right)\right]\right) : \psi \in \mathbb{R}\right\} \cong \mathrm{SO}(2), & \lambda > 0, \\ \{([U^\dagger], [U]) : [U] \in PSU(2)\} \cong \mathrm{SO}(3), & \lambda = 0. \end{cases} \tag{3.10}$$

The induced action of H_λ on $V_\lambda^* \otimes V_\lambda^*$ leaves the subspaces of symmetric and antisymmetric bilinear forms invariant, that is, preserves the splitting

$$V_\lambda^* \otimes V_\lambda^* = [V_\lambda^* \odot V_\lambda^*] \oplus [V_\lambda^* \wedge V_\lambda^*] =: V_\lambda^+ \oplus V_\lambda^-. \tag{3.11}$$

One may compute the dimension of the subspace of V_λ^+ on which H_λ acts trivially (i.e. the subspace of H_λ invariant symmetric bilinear forms) by counting the number of copies of the trivial representation in the decomposition of (H_λ, V_λ^+) into irreducible representations, using character orthogonality. Eq. (3.8) captures all possibilities if and only if this dimension is 7 for $\lambda > 0$ and 3 for $\lambda = 0$.

Consider first the generic case, $\lambda > 0$, $H_\lambda \cong \mathrm{SO}(2)$. The H_λ action on V_λ has matrix representation

$$R(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 & 0 & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\psi & \sin\psi & 0 \\ 0 & 0 & 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.12}$$

relative to the ordered basis $(\partial/\partial\lambda_1, \dots, \theta_3)$, where $\{\theta_a\}$ are the left-invariant vector fields dual to $\{\sigma_a\}$. Hence the character $\chi : H_\lambda \to \mathbb{R}$ of this representation is

$$\chi(\psi) = \mathrm{tr}\, R(\psi) = 2 + 4\cos\psi. \tag{3.13}$$

The character of the induced representation of SO(2) on $V_\lambda^\pm$ is [11]

$$\begin{aligned} \tilde{\chi}_\pm(\psi) &= \frac{1}{2}\{[\mathrm{tr}\, R(\psi)]^2 \pm \mathrm{tr}[R(\psi)^2]\} \\ &= \begin{cases} 7 + 8\cos\psi + 6\cos 2\psi, & \text{symmetric}, \\ 5 + 8\cos\psi + 2\cos 2\psi, & \text{antisymmetric}. \end{cases} \end{aligned} \tag{3.14}$$

We shall make use of the result for V_λ^- when analysing the J-associated 2-form $\hat{\tau}$. Since $\mathrm{SO}(N)$ is a compact Lie group, the characters of inequivalent irreducible representations are orthogonal functions on $\mathrm{SO}(N)$ with respect to the Haar measure. One may therefore extract the coefficient $a_0^\pm$ of the trivial character ($\chi_0(\psi) = 1$) from the decomposition

$$\tilde{\chi}_\pm = \sum_n a_n^\pm \chi_n \tag{3.15}$$

of $\tilde{\chi}_\pm$ into irreducible representations by taking the character inner product of both sides of (3.15) with χ_0

$$a_0^\pm \int_{SO(N)} d\mu\, \chi_0^2 = \int_{SO(N)} d\mu\, \chi_0 \tilde{\chi}_\pm, \tag{3.16}$$

where $d\mu$ is the Haar measure. For SO(2), $d\mu = d\psi/2\pi$, so

$$a_0^\pm = \int_0^{2\pi} \frac{d\psi}{2\pi} \tilde{\chi}_\pm(\psi) = \begin{cases} 7, & \text{symmetric}, \\ 5, & \text{antisymmetric}, \end{cases} \tag{3.17}$$

in agreement with (3.8).

In the special case $\lambda = 0$, the isotropy group is $H_0 \cong SO(3)$ whose action on V_λ has matrix representation

$$R(\psi, \hat{\boldsymbol{n}}) = \begin{pmatrix} \mathcal{O}(\psi, \hat{\boldsymbol{n}}) & 0 \\ 0 & \mathcal{O}(\psi, \hat{\boldsymbol{n}}) \end{pmatrix}, \tag{3.18}$$

where $(\psi, \hat{\boldsymbol{n}})$ parameterizes the rotation through angle ψ about axis $\hat{\boldsymbol{n}} \in S^2$ and $\mathcal{O}(\psi, \hat{\boldsymbol{n}})$ is the associated SO(3) matrix. The character of this representation is

$$\chi(\psi, \hat{\boldsymbol{n}}) = 2\,\mathrm{tr}\,\mathcal{O}(\psi, \hat{\boldsymbol{n}}) = 2(1 + e^{i\psi} + e^{-i\psi}) = 2 + 4\cos\psi. \tag{3.19}$$

It follows from (3.13), (3.14) and (3.19) that the characters of the induced representations on $V_\lambda^\pm$ are the same trigonometric functions $\tilde{\chi}_\pm(\psi)$ above, independent of $\hat{\boldsymbol{n}}$. Once again, we may extract $a_0^\pm$ using character orthogonality, but now we must integrate over SO(3) using the Haar measure, which is

$$d\mu = \frac{1}{\pi} \sin^2\frac{\psi}{2}\, d\psi, \tag{3.20}$$

after integrating over $\hat{\boldsymbol{n}}$ [12]. The result is

$$a_0^\pm = \frac{1}{\pi} \int_0^{2\pi} d\psi \sin^2\frac{\psi}{2} \tilde{\chi}_\pm(\psi) = \begin{cases} 3, & \text{symmetric}, \\ 1, & \text{antisymmetric}, \end{cases} \tag{3.21}$$

which proves the initial claim.

Since τ is G invariant (not merely G_0 invariant), it must also be invariant under the discrete isometry P, which in matrix terms is $P : [M] \mapsto [\bar{M}]$ (entrywise complex conjugation). The pull-back action on the moving coframe is

$$P^* : (d\boldsymbol{\lambda}, \boldsymbol{\sigma}) \mapsto (d\lambda_1, -d\lambda_2, d\lambda_3, -\sigma_1, \sigma_2, -\sigma_3), \tag{3.22}$$

implying that $A_6 \equiv A_7 \equiv 0$.

It remains to show that the coefficient functions $A_1, \ldots, A_5$ are determined by the single function A as in (3.7). This follows from Hermiticity of τ and closure of $\hat{\tau}$. Recall that the complex structure on M_1 is inherited from the open inclusion $M_1 \subset \mathbb{C}P^3$. For example, on the open set where $a_{11} \neq 0$ (Eq. (3.1)), we may use the inhomogeneous coordinates

$$b_1 = \frac{a_{12}}{a_{11}}, \qquad b_2 = \frac{a_{21}}{a_{11}}, \qquad b_3 = \frac{a_{22}}{a_{11}}, \tag{3.23}$$

to define a complex coordinate chart. This chart contains the curve Γ we are using to parameterize the orbit space M_1/G_0. It is a simple matter to write down the almost complex structure J-associated with this complex structure, in terms of the basis $\{\partial/\partial\lambda_a, \theta_a : a = 1,2,3\}$ for V_λ, namely

$$J : \frac{\partial}{\partial\lambda_1} \mapsto \frac{2}{\Lambda}\left(\theta_1 - \frac{\lambda}{2}\frac{\partial}{\partial\lambda_2}\right), \qquad J : \frac{\partial}{\partial\lambda_2} \mapsto \frac{2}{\Lambda}\left(\theta_2 + \frac{\lambda}{2}\frac{\partial}{\partial\lambda_1}\right),$$
$$J : \frac{\partial}{\partial\lambda_3} \mapsto \frac{2}{\Lambda}\theta_3, \qquad J : \theta_1 \mapsto -\frac{1}{2\Lambda}\left(\frac{\partial}{\partial\lambda_1} - 2\lambda\theta_2\right),$$
$$J : \theta_2 \mapsto -\frac{1}{2\Lambda}\left(\frac{\partial}{\partial\lambda_2} + 2\lambda\theta_1\right), \qquad J : \theta_3 \mapsto -\frac{\Lambda}{2}\frac{\partial}{\partial\lambda_3}. \tag{3.24}$$

We emphasize that (3.24) is valid only on tangent spaces based at points on the curve Γ. By G invariance of τ, this will be all the information we need.

Hermiticity of τ, $\tau_\lambda(JX, JY) \equiv \tau_\lambda(X, Y)$ for all $X, Y \in V_\lambda$, produces two nontrivial constraints on the coefficients $A_1, \ldots, A_5$, namely,

$$A_3 \equiv \frac{A_1}{4} + \frac{\lambda^2}{2}A_5, \qquad A_1 + \lambda^2 A_2 \equiv \frac{4}{1+\lambda^2}(A_3 + \lambda^2 A_4). \tag{3.25}$$

Let $f \in G$, and denote by the same symbol its action on M_1, $f : \mathrm{M}_1 \to \mathrm{M}_1$. The 2-form $\hat{\tau}(\cdot,\cdot) = \tau(J\cdot,\cdot)$ is invariant, $f^*\hat{\tau} = \hat{\tau}$, under any holomorphic $f \in G$ since $f^*\tau = \tau$ (G invariance of τ) and $\mathrm{d}f_W \circ J_W = J_{f(W)} \circ \mathrm{d}f_W$ (holomorphicity). Similarly, $f^*\hat{\tau} = -\hat{\tau}$ for antiholomorphic $f \in G$. Now each $f \in G_0$ is holomorphic, so $\hat{\tau}$ is G_0 invariant. We claim that the most general G_0 invariant 2-form on M_1 is

$$\hat{\tau} = \hat{A}_1(\mathrm{d}\boldsymbol{\lambda}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\mathrm{d}\boldsymbol{\lambda}) + \hat{A}_2((\boldsymbol{\lambda}\cdot\mathrm{d}\boldsymbol{\lambda})(\boldsymbol{\lambda}\cdot\boldsymbol{\sigma}) - (\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})(\boldsymbol{\lambda}\cdot\mathrm{d}\boldsymbol{\lambda})) + \hat{A}_3\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\boldsymbol{\sigma})$$
$$+ \hat{A}_4\boldsymbol{\lambda}\cdot(\mathrm{d}\boldsymbol{\lambda}\times\mathrm{d}\boldsymbol{\lambda}) + \hat{A}_5(\mathrm{d}\boldsymbol{\lambda}\cdot(\boldsymbol{\lambda}\times\boldsymbol{\sigma}) - (\boldsymbol{\lambda}\times\boldsymbol{\sigma})\cdot\mathrm{d}\boldsymbol{\lambda}), \tag{3.26}$$

where $\hat{A}_1, \ldots, \hat{A}_5$ are functions of λ only, and juxtaposition of 1-forms indicates *unsymmetrized* tensor product. Clearly, such a 2-form is G_0 invariant by (3.9), and is the most general such form possible by (3.17) and (3.21). In fact, since $P : [M] \mapsto [\bar{M}]$ is antiholomorphic, $P^*\hat{\tau} = -\hat{\tau}$, and we may immediately conclude that $\hat{A}_5 \equiv 0$.

It is a simple matter to match $\hat{\tau}(\cdot,\cdot)$ with $\tau_\lambda(J\cdot,\cdot)$ on V_λ using (3.24), and hence determine $\hat{A}_1, \ldots, \hat{A}_4$ in terms of $A_1, \ldots, A_5$. The result is

$$\hat{A}_1 = \frac{\Lambda}{2}A_1, \qquad \hat{A}_2 = \frac{\Lambda}{2}A_2, \qquad \hat{A}_3 = \frac{1}{4\Lambda}(A_1 + 4A_3), \qquad \hat{A}_4 = \frac{\lambda}{\Lambda}(A_5 - A_1). \tag{3.27}$$

Closure of $\hat{\tau}$ then gives extra constraints on the metric coefficients $A_1, \ldots, A_5$. Using the standard exterior differential algebra for the left-invariant 1-forms of SO(3),

$$\mathrm{d}\sigma_1 = \sigma_2 \wedge \sigma_3, \qquad \mathrm{d}\sigma_2 = \sigma_3 \wedge \sigma_1, \qquad \mathrm{d}\sigma_3 = \sigma_1 \wedge \sigma_2, \tag{3.28}$$

one finds that at any $W_\lambda \in \Gamma$,

$$\begin{aligned}\mathrm{d}\hat{\tau} = {} & (\hat{A}_1' - \lambda \hat{A}_2)\,\mathrm{d}\lambda_3 \wedge (\mathrm{d}\lambda_1 \wedge \sigma_1 + \mathrm{d}\lambda_2 \wedge \sigma_2) \\ & + (\hat{A}_3 - \hat{A}_1)(\mathrm{d}\lambda_1 \wedge \sigma_2 \wedge \sigma 3 + \mathrm{d}\lambda_2 \wedge \sigma_3 \wedge \sigma 1 + \mathrm{d}\lambda_3 \wedge \sigma_1 \wedge \sigma 2) \\ & + \lambda(\hat{A}_3' - \lambda \hat{A}_2)\,\mathrm{d}\lambda_3 \wedge \sigma_1 \wedge \sigma_2 + (\lambda \hat{A}_4' + 3\hat{A}_4)\,\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3. \end{aligned} \quad (3.29)$$

Hence, $\mathrm{d}\hat{\tau} = 0$ if and only if

$$\hat{A}_1 = \hat{A}_3, \qquad \hat{A}_1' = \lambda \hat{A}_2, \qquad \hat{A}_4 = 0, \quad (3.30)$$

the last of these following from nonsingularity of $\hat{\tau}$ at $\lambda = 0$. Rearranging these using (3.27) and the Hermiticity constraints (3.25), one finds that all the metric coefficients are determined by the single smooth function $A_1 = A(\lambda)$ as in (3.7). □

Corollary 3.2. *The L^2 metric on* M_1 *is*

$$\gamma = A_1\,\mathrm{d}\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda} + A_2(\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda})^2 + A_3 \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} + A_4(\boldsymbol{\lambda} \cdot \boldsymbol{\sigma})^2 + A_5 \boldsymbol{\lambda} \cdot (\boldsymbol{\sigma} \times \mathrm{d}\boldsymbol{\lambda}),$$

where $A_1, \ldots, A_5$ are functions of λ only, determined as in (3.7) *by the single function*

$$A = \frac{4\pi\mu[\mu^4 - 4\mu^2 \log\mu - 1]}{(\mu^2 - 1)^3}, \quad (3.31)$$

where $\mu = (\sqrt{1+\lambda^2} + \lambda)^2$.

Proof. By Theorem 2.1, γ is Hermitian and its J-associated 2-form (the Kähler form, henceforth denoted Ω, rather than $\hat{\gamma}$) is closed. Furthermore, γ is G invariant. Hence Proposition 3.1 applies. The formula for A is obtained by computing $\gamma_\lambda(\partial/\partial\lambda_1, \partial/\partial\lambda_1)$ using (3.7). □

Given a tensor τ satisfying the hypotheses of Proposition 3.1, it is convenient to define a second coefficient function, $B(\lambda) := \tau_\lambda(\theta_3, \theta_3)$. Of course, B is determined by A, according to (3.7)

$$B(\lambda) = A_3 + \lambda^2 A_4 \equiv \frac{1+2\lambda^2}{4} A(\lambda) + \frac{\lambda + \lambda^3}{4} A'(\lambda). \quad (3.32)$$

One finds for $\tau = \gamma$, the L^2 metric, that

$$B = \frac{4\pi\mu^2[(\mu^2+1)\log\mu - \mu^2 + 1]}{(\mu^2 - 1)^3}. \quad (3.33)$$

An explicit formula for γ has previously appeared in the physics literature [32], although its Kähler property and the resulting interdependence of the coefficient functions was not understood, nor was a rigorous classification of G invariant tensors on M_1 performed. The geodesic flow on (M_1, γ) has been extensively studied, also in [32], revealing quite complicated lump dynamics. We finish this section by examining the large λ behaviour of γ. Specifically, we will prove that (M_1, γ) has finite volume and diameter, and describe its boundary at infinity.

Theorem 3.3. (M_1, γ) *has finite volume and diameter*.

Proof. The volume form is

$$\mathrm{Vol} = \frac{\Lambda}{2} BA^2\, \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3. \tag{3.34}$$

Hence,

$$\begin{aligned}\mathrm{Vol}(\mathrm{M}_1, \gamma) &= 4\pi\, \mathrm{Vol}(\mathrm{SO}(3)) \int_0^\infty \mathrm{d}\lambda\, \lambda^2 \frac{\sqrt{1+\lambda^2}}{2} BA^2 \\ &= \frac{\pi}{16} \mathrm{Vol}(\mathrm{SO}(3)) \int_1^\infty \frac{\mathrm{d}\mu}{\mu} \left(\mu - \frac{1}{\mu}\right)^2 BA^2 \\ &< c + \pi^3\, \mathrm{Vol}(\mathrm{SO}(3)) \int_2^\infty \mathrm{d}\mu\, \mu \left(2^4 \frac{\log \mu}{\mu^2}\right) \left(\frac{2^3}{\mu}\right)^2,\end{aligned} \tag{3.35}$$

where c is a constant (the volume from $\mu = 1$ to 2). Hence (M_1, γ) has finite volume.

One may similarly bound the diameter of (M_1, γ),

$$\mathrm{diam}(\mathrm{M}_1, \gamma) := \sup_{W_1, W_2 \in \mathrm{M}_1} \mathrm{d}(W_1, W_2). \tag{3.36}$$

By the triangle inequality,

$$\mathrm{diam}(\mathrm{M}_1, \gamma) \leq 2 \sup_{W \in \mathrm{M}_1} \mathrm{d}(W, \mathrm{Id}). \tag{3.37}$$

The distance of any map W from Id is bounded above by the sum of the length of the radial curve from $([U], \boldsymbol{\lambda})$ to $([U], \mathbf{0})$ and the distance in SO(3) from $[U]$ to $[\mathbb{I}]$ with respect to the bi-invariant metric $A_3(0)\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}$. The latter contribution is bounded independent of $[U]$ by compactness of SO(3), and the former is, by G_0 invariance, bounded above by the length of the curve Γ. But

$$\begin{aligned}\mathrm{length}(\Gamma) &= \int_0^\infty \mathrm{d}\lambda \sqrt{A_1 + \lambda^2 A_2} \\ &= \int_1^\infty \frac{\mathrm{d}\mu}{\mu} \sqrt{B} < c + 8\sqrt{\pi} \int_2^\infty \mathrm{d}\mu\, \frac{\sqrt{\log \mu}}{\mu^2} < \infty.\end{aligned} \tag{3.38}$$

Hence (M_1, γ) has finite diameter. □

For both estimates, the key point is that $A(\lambda)$ and $A'(\lambda)$ decay sufficiently rapidly as $\lambda \to \infty$ to guarantee convergence of the integrals. Note that while every G invariant Kähler metric on M_1 is determined by a single function $A(\lambda)$, the converse is false: not every $A(\lambda)$ defines such a metric since one must also demand that γ be positive definite. This places one nontrivial constraint on A

$$\gamma_\lambda \left(\frac{\partial}{\partial \lambda_3}, \frac{\partial}{\partial \lambda_3}\right) > 0 \Rightarrow \frac{A'}{A} > -\frac{1+2\lambda^2}{\lambda + \lambda^3}, \tag{3.39}$$

and one trivial constraint ($A > 0$), which together bound the decay rate of $A(\lambda)$ as $\lambda \to \infty$. Integrating inequality (3.39) yields, for example,

$$A(\lambda) > \frac{\sqrt{2} A(1)}{\lambda \sqrt{1+\lambda^2}} \quad \forall \lambda > 1, \tag{3.40}$$

so the decay of A cannot be faster than $\mathrm{O}(1/\lambda\Lambda)$. It is interesting to note that the asymptotic behaviour of the L^2 metric saturates this bound, namely $\lim_{\lambda\to\infty}\lambda\Lambda A = \pi$.

As shown above, the boundary of (M_1, γ) at infinity lies at finite proper distance, so the space is geodesically incomplete. One expects, however, that generic geodesics do *not* escape to infinity, since the boundary has codimension 2, as we now show.

Theorem 3.4. *The boundary at infinity of* (M_1, γ) *is diffeomorphic to* $S^2 \times S^2$.

Proof. The idea is to analyse the 1-parameter family of homogeneous metrics on $\mathrm{SO}(3)\times S^2$ induced by γ, using the cohomogeneity 1 property. Consider the pullback by the left coset projection $\pi_\lambda : G \to G/H(\lambda) \hookrightarrow \mathrm{M}_1$ of the metric γ. The orbit itself (level set of $\lambda \in [0,\infty)$) may be identified with the quotient of G by the subgroup generated by the null space of $(\pi_\lambda^*\gamma)_{(\mathbb{I},\mathbb{I})}$ (the isotropy group $H(\lambda)$ of W_λ by nondegeneracy of γ). With respect to the basis $(\theta_i, 0)$, $(\theta_1, -\theta_1)$, $i = 1, 2, 3$ for $T_{(\mathbb{I},\mathbb{I})}G$, this bilinear form has coefficient matrix

$$[(\pi_\lambda^*\gamma)_{(\mathbb{I},\mathbb{I})}] = \begin{pmatrix} \frac{1}{4}(\Lambda^2+\lambda^2)A & 0 & 0 & 0 & \frac{1}{2}\lambda A & 0 \\ 0 & \frac{1}{4}(\Lambda^2+\lambda^2)A & 0 & -\frac{1}{2}\lambda A & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\lambda A & 0 & \lambda^2 A & 0 & 0 \\ \frac{1}{2}\lambda A & 0 & 0 & 0 & \lambda^2 A & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We seek to construct the "$\lambda = \infty$" orbit. As $\lambda \to \infty$, the matrix above converges to

$$\mathrm{diag}\left(\frac{\pi}{2}, \frac{\pi}{2}, 0, \pi, \pi, 0\right),$$

whose null space generates the toric subgroup

$$T^2 = \left\{\left(\left[\exp\left(\frac{\mathrm{i}}{2}\alpha\tau_3\right)\right], \left[\exp\left(\frac{\mathrm{i}}{2}\beta\tau_3\right)\right]\right) : \alpha, \beta \in \mathbb{R}\right\}.$$

Hence $\partial(\mathrm{M}_1, \gamma) \cong G/T^2 \cong S^2 \times S^2$. □

Remark 3.5. The identification $\partial(\mathrm{M}_1, \gamma) \cong S^2 \times S^2$ is natural in two senses. First, the set of pointwise limit maps $\phi_\infty : S^2 \to S^2$, obtained by taking the $\lambda \to \infty$ limit of the $([U], \boldsymbol{\lambda})$ rational map, is naturally in bijective correspondence with $S^2 \times S^2$. To specify such a limit map one must choose a point p in the codomain, to which almost every point in the domain is mapped, and a point in the domain, which alone is mapped to the antipodal point to p.

Second, the complex codimension 1 algebraic variety complementary to Rat_1 in $\mathbb{C}P^3$, as described in Section 2, is diffeomorphic to $S^2 \times S^2$. Indeed, it is the image of the holomorphic embedding $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3$, $([x_1, x_2], [y_1, y_2]) \to [x_1y_2, x_1y_1, x_2y_2, x_2y_1]$.

4. Curvature properties

4.1. Holomorphic sectional curvature

Recall that the sectional curvature of a plane $P \in \mathrm{Gr}_2(T_W\mathrm{M}_1)$ is

$$\sigma(X, Y) := \langle R(X, Y)Y, X\rangle, \tag{4.1}$$

where X, Y are orthonormal and span P, $\langle\cdot,\cdot\rangle = \gamma(\cdot,\cdot)$ and R is the Riemann curvature tensor [38]. Recall also that, since γ is Hermitian, $\gamma(X, JX) \equiv 0$ and $\|JX\| \equiv \|X\|$, so one may assign to a line $L \in \mathrm{Gr}_1(T_W\mathrm{M}_1)$ containing X, $\|X\| = 1$, the holomorphic sectional curvature

$$\mathrm{Hol}(X) := \sigma(X, JX). \tag{4.2}$$

In fact, given that γ is Kähler, Hol uniquely determines σ and hence R [13].

We shall compute the holomorphic sectional curvature of the unitary frame $\{e_a, Je_a : a = 1, 2, 3\}$ for V_λ, where

$$e_1 = \frac{1}{\sqrt{A_1}}\frac{\partial}{\partial\lambda_1}, \qquad e_2 = \frac{1}{\sqrt{A_1}}\frac{\partial}{\partial\lambda_2}, \qquad e_3 = \frac{1}{\sqrt{A_1+\lambda^2 A_2}}\frac{\partial}{\partial\lambda_3}. \tag{4.3}$$

Hermiticity implies that $\mathrm{Hol}(X) \equiv \mathrm{Hol}(JX)$, and G invariance implies that $\mathrm{Hol}(e_1) \equiv \mathrm{Hol}(e_2)$, so we shall calculate only $\mathrm{Hol}(e_1)$ and $\mathrm{Hol}(e_3)$. These will vary with basepoint $W_\lambda \in \Gamma$, and hence be functions of λ.

The simpler of the two is $\mathrm{Hol}(e_3)$:

$$\begin{aligned}
\mathrm{Hol}(e_3) &= \frac{4}{(1+\lambda^2)(A_1+\lambda^2A_2)^2}\left\langle \nabla_{\partial/\partial\lambda_3}\nabla_{\theta_3}\theta_3 - \nabla_{\theta_3}\nabla_{\partial/\partial\lambda_3}\theta_3 - \nabla_{[\partial/\partial\lambda_3,\theta_3]}\theta_3, \frac{\partial}{\partial\lambda_3}\right\rangle \\
&= \frac{4}{(1+\lambda^2)(A_1+\lambda^2A_2)^2} \\
&\quad \times \left\{\frac{\partial}{\partial\lambda_3}\left\langle\nabla_{\theta_3}\theta_3, \frac{\partial}{\partial\lambda_3}\right\rangle - \left\langle\nabla_{\theta_3}\theta_3, \nabla_{\partial/\partial\lambda_3}\frac{\partial}{\partial\lambda_3}\right\rangle + \|\nabla_{\partial/\partial\lambda_3}\theta_3\|^2\right\} \\
&= \frac{1+\lambda^2}{8B^2}\left\{\left(\frac{B'}{B} - \frac{\lambda}{1+\lambda^2}\right)B' - B''\right\}.
\end{aligned} \tag{4.4}$$

To obtain (4.4), we have used metric compatibility and torsionlessness of ∇, left SO(3) invariance of γ and the Lie algebra $\mathrm{su}(2)\oplus\mathbb{R}^3$, namely,

$$\left[\frac{\partial}{\partial\lambda_a}, \frac{\partial}{\partial\lambda_b}\right] = \left[\frac{\partial}{\partial\lambda_a}, \theta_b\right] = 0, \quad [\theta_a, \theta_b] = -\epsilon_{abc}\theta_c. \tag{4.5}$$

Formula (4.4) may be written in terms of A alone using (3.32), but the result is rather messy.

Due to the more complicated expression for Je_1, in comparison with Je_3 (see (3.24)), the calculation of $\mathrm{Hol}(e_1)$ is considerably lengthier, though no more technically difficult. We merely record the result, which, unlike $\mathrm{Hol}(e_3)$, simplifies somewhat when expressed

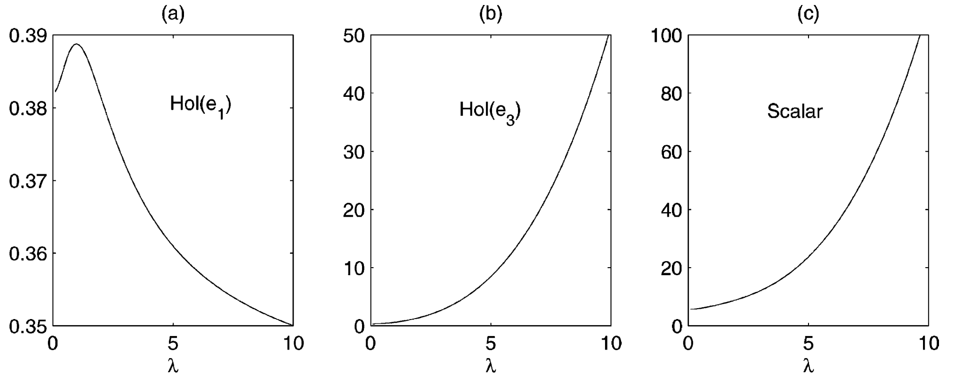

Fig. 1. Plots of various curvature functions against the radial coordinate λ for the L^2 metric on M_1. Note the unboundedness of $\mathrm{Hol}(e_3)$ and κ (scalar curvature).

purely in terms of A:

$$\mathrm{Hol}(e_1) = \frac{1}{A^2\Lambda^2}\left\{\frac{\lambda A + \frac{1}{2}\Lambda^2 A'}{(\Lambda^2+\lambda^2)A + \lambda\Lambda^2 A'}\left(\frac{\lambda A}{\Lambda^2} + A'\right) - \frac{2+\lambda^2}{1+\lambda^2}A - \frac{3+2\lambda^2}{2\lambda}A'\right\}. \tag{4.6}$$

Substituting formulae (3.31) and (3.33) for $A(\lambda)$ and $B(\lambda)$ into (4.4) and (4.6), one obtains (very complicated) explicit expressions for $\mathrm{Hol}(e_3)$ and $\mathrm{Hol}(e_1)$. Plots of these are presented in Fig. 1. Note that, although $\mathrm{Hol}(e_1)$ is bounded, $\mathrm{Hol}(e_3)$ is unbounded above. In fact, one finds (using Maple, for example) that

$$\lim_{\lambda\to\infty}\mathrm{Hol}(e_1) = \frac{1}{\pi}, \qquad \lim_{\lambda\to\infty}\frac{(\log\lambda)^3}{\lambda^4}\mathrm{Hol}(e_3) = \frac{1}{4\pi}, \tag{4.7}$$

which proves the following theorem.

Theorem 4.1. *The holomorphic sectional curvature of* (M_1, γ) *is unbounded above. Hence, no isometric compactification of* (M_1, γ) *exists, despite its finite volume and diameter.*

4.2. Ricci curvature

Recall that the Ricci curvature ρ of a Riemannian manifold is the symmetric (0, 2) tensor

$$\rho(X, Y) := \mathrm{tr}(V \mapsto R(V, X)Y), \tag{4.8}$$

where R is the Riemann curvature tensor, as before [14].

Proposition 4.2. *Let* γ *be a* G *invariant Kähler metric on* M_1, *determined as in* Proposition 3.1 *by the function* A. *Then the Ricci curvature of* (M_1, γ) *is*

$$\rho = \bar{A}_1\,\mathrm{d}\boldsymbol{\lambda}\cdot\mathrm{d}\boldsymbol{\lambda} + \bar{A}_2(\boldsymbol{\lambda}\cdot\mathrm{d}\boldsymbol{\lambda})^2 + \bar{A}_3\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + \bar{A}_4(\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})^2 + \bar{A}_5\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\mathrm{d}\boldsymbol{\lambda}), \tag{4.9}$$

where $\bar{A}_1, \ldots, \bar{A}_5$ *are functions of* λ *only, determined as in* (3.7) *by the single function*

$$\bar{A} = -\frac{2\lambda(1+\lambda^2)(A')^2 + (9\lambda^2+4)AA' + \lambda(1+\lambda^2)AA'' + 4A^2\lambda}{2\lambda A(A + 2\lambda^2 A + \lambda A' + \lambda^3 A')}. \tag{4.10}$$

Proof. Since the G action is isometric, ρ is G invariant. Furthermore, since γ is Kähler, $\rho(JX, JY) \equiv \rho(X, Y)$ [15], and the associated Ricci form, $\hat{\rho}$ is closed [16]. Hence, Proposition 3.1 applies to ρ just as it applies to γ, and all the coefficient functions are determined by $\rho_\lambda(\partial/\partial\lambda_1, \partial/\partial\lambda_1) = \bar{A}(\lambda)$. But $\rho_\lambda(\partial/\partial\lambda_1, \partial/\partial\lambda_1)$ is determined by A according to Eq. (4.8), which yields formula (4.10). □

As with the metric, it is convenient to define the associated coefficient function

$$\bar{B}(\lambda) := \rho_\lambda(\theta_3, \theta_3) = \bar{A}_3 + \lambda^2 \bar{A}_4 = \frac{1+2\lambda^2}{4}\bar{A}(\lambda) + \frac{\lambda+\lambda^3}{4}\bar{A}'(\lambda). \tag{4.11}$$

An explicit formula for the Ricci curvature of the L^2 metric is obtained by substituting (3.31) into (4.10). Unfortunately, this formula is far too complicated to be instructive. However, it leads us to the following conjecture.

Conjecture 4.3. *The Ricci curvature of the* L^2 *metric on* M_1 *is positive definite.*

In support of this, note that, relative to the ordered basis $(\partial/\partial\lambda_1, \theta_2, \partial/\partial\lambda_2, \theta_1, \partial/\partial\lambda_3, \theta_3)$, the coefficient matrix of ρ_λ is block diagonal with blocks

$$\bar{A}\begin{pmatrix} 1 & -\dfrac{\lambda}{2} \\ -\dfrac{\lambda}{2} & \dfrac{1+2\lambda^2}{4} \end{pmatrix}, \qquad \bar{A}\begin{pmatrix} 1 & \dfrac{\lambda}{2} \\ \dfrac{\lambda}{2} & \dfrac{1+2\lambda^2}{4} \end{pmatrix}, \qquad \bar{B}\begin{pmatrix} \dfrac{4}{1+\lambda^2} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.12}$$

whence it follows that ρ_λ is positive definite if and only if $\bar{A}(\lambda) > 0$ and $\bar{B}(\lambda) > 0$. Now $\bar{A}(0) = 4$ and $\bar{B}(0) = 1$, so ρ is certainly positive definite in a neighbourhood of Id, and

$$\lim_{\lambda\to\infty} \lambda^2 \bar{A}(\lambda) = 4, \qquad \lim_{\lambda\to\infty} (\log\lambda)^2 \bar{B}(\lambda) = \tfrac{1}{8}, \tag{4.13}$$

so ρ is asymptotically positive definite also. Convincing graphical evidence in favour of the conjecture is presented in Fig. 2, which contains plots of $\bar{A}$ and $\bar{B}$.

We note in passing that the Einstein field equations for G invariant Kähler metrics

$$\rho = \tfrac{1}{6}(\kappa)\gamma \tag{4.14}$$

reduce to a single second order nonlinear ODE, explicit solutions to which may be constructed in the Ricci flat case. The results will be described in detail elsewhere.

4.3. Scalar curvature

While Hol and ρ are not directly relevant to soliton dynamics, the scalar curvature κ certainly is, at least in the quantum regime. The standard approach to low energy quantum

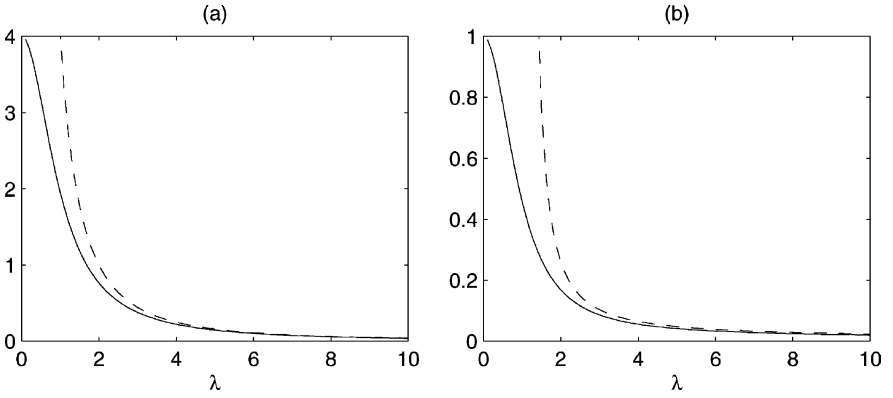

Fig. 2. Plots of the coefficient functions of the Ricci curvature of γ: (a) $\bar{A}(\lambda)$ and (b) $\bar{B}(\lambda)$. Note that both are positive within the plot domain, and that for $\lambda \geq 4$, they are very close to the asymptotic forms $4\lambda^{-2}$ and $[(\log\lambda)^{-2}]/8$, respectively (the dashed curves).

n-soliton dynamics [8] is to assume that the quantum state is well described by a wavefunction on the n-soliton moduli space $\psi : \mathrm{M}_n \to \mathbb{C}$ (which receives the usual probabilistic interpretation) subject to a Schrödinger equation of the form

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta_\gamma\psi + V\psi, \tag{4.15}$$

where Δ_γ is the covariant Laplacian on (M_n, γ) and $V : \mathrm{M}_n \to \mathbb{R}$ is a potential function. The question of precisely what terms should be included in V is somewhat controversial, and the answer likely varies according to exact context. However, there seems to be general agreement that, following De Witt [5], one should include (a positive multiple of) κ in V. For a recent discussion of this subject, specifically in the context of σ-models, see [23]. So the relevance of κ to quantum lump dynamics, as well as simple geometric curiosity, motivate us to calculate it.

Proposition 4.4. *Let γ be a G invariant Kähler metric on* M_1, *determined as in Proposition 3.1 by the function $A(\lambda)$. Then the scalar curvature of* (M_1, γ) *is*

$$\kappa = 4\frac{\bar{A}}{A} + 2\frac{\bar{B}}{B}, \tag{4.16}$$

where $\bar{A}$ and B are determined by A as in Eqs. (4.10) and (3.32), and $\bar{B}$ is determined by $\bar{A}$ as in Eq. (4.11).

Proof. By G invariance, κ is a function of λ only, so it suffices to compute it at $W_\lambda \in \Gamma$. Making use of the unitary frame $\{e_a, Je_a : a = 1, 2, 3\}$ and recalling that $\rho(JX, JY) \equiv \rho(X, Y)$, one finds

$$\kappa = 2\sum_{a=1}^{3}\rho(e_a, e_a) = 2\left[2\frac{\bar{A}_1}{A_1} + \frac{\bar{A}_1 + \lambda^2\bar{A}_2}{A_1 + \lambda_2 A_2}\right], \tag{4.17}$$

in the notation of Proposition 4.2. Formula (4.16) follows from applying the relations (3.25), (3.32), (4.11)–(4.17). □

Corollary 4.5. *The scalar curvature of the L^2 metric on* M_1 *is unbounded above.*

Proof. From Eqs. (3.31) and (3.33) one has the limits

$$\lim_{\lambda\to\infty} \lambda^2 A(\lambda) = \pi, \qquad \lim_{\lambda\to\infty} \frac{\lambda^4}{\log\lambda} B(\lambda) = \frac{\pi}{2}, \tag{4.18}$$

which together with (4.13) and Proposition 4.4 imply that

$$\lim_{\lambda\to\infty} \frac{(\log\lambda)^3}{\lambda^4}\kappa(\lambda) = \frac{1}{2\pi}. \qquad \square \tag{4.19}$$

Remark 4.6. Numerical evidence suggests that the L^2 metric on M_1 has strictly positive scalar curvature (see Fig. 1), as one would expect, given Conjecture 4.3.

Since (M_1, γ) is noncompact, but of finite volume, the question of what boundary conditions to impose on the quantum wavefunction ψ at $\lambda = \infty$ when seeking bound states is nontrivial. The fact that $\kappa \to \infty$ as $\lambda \to \infty$ supports the imposition of vanishing boundary conditions for all quantum states of finite energy. One would expect the quantum 1-lump energy spectrum to be discrete, therefore.

4.4. The Fubini–Study metric

There is another natural Kähler metric on M_1 given by the open inclusion $\mathrm{M}_1 \subset \mathbb{C}P^3$, namely the Fubini–Study metric on $\mathbb{C}P^3$. In terms of the local inhomogeneous coordinates b_1, b_2, b_3 (3.23) this takes the form [39]

$$\gamma_{\mathrm{FS}} = \frac{(1+\sum|b_a|^2)(1+\sum \mathrm{d}b_b\,\overline{\mathrm{d}b_b}) - (\sum \bar{b}_a\,\mathrm{d}b_a)(\sum b_b\,\overline{\mathrm{d}b_b})}{(1+\sum|b_a|^2)^2}. \tag{4.20}$$

Proposition 4.7. *The Fubini–Study metric on* M_1 *is*

$$\gamma_{\mathrm{FS}} = A_1\,\mathrm{d}\boldsymbol{\lambda}\cdot\mathrm{d}\boldsymbol{\lambda} + A_2(\boldsymbol{\lambda}\cdot\mathrm{d}\boldsymbol{\lambda})^2 + A_3\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + A_4(\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})^2 + A_5\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\mathrm{d}\boldsymbol{\lambda}),$$

$A_1, \ldots, A_5$ *being determined as in* (3.7) *by the single function*

$$A_{\mathrm{FS}}(\lambda) = \frac{2\mu(\lambda)}{1+\mu(\lambda)^2}, \tag{4.21}$$

where $\mu(\lambda) = (\sqrt{1+\lambda^2}+\lambda)^2$.

Proof. The isometric action of PU(4) on $(\mathbb{C}P^3, \gamma_{\mathrm{FS}})$ obtained by projecting the standard $U(4)$ action on $\mathbb{C}^4$ contains the G_0 action on M_1 we have been considering. Furthermore, γ_{FS} is manifestly invariant under $M \mapsto \bar{M}$ (i.e. $b_a \mapsto \bar{b}_a$) from (4.20). Hence Proposition 3.1 applies. It remains to compute $A_{\mathrm{FS}}(\lambda) = \gamma_{\mathrm{FS}}(\partial/\partial\lambda_1, \partial/\partial\lambda_1)$ at $W_\lambda \in \Gamma$, using (4.20), which is straightforward algebra. □

Proposition 4.7 gives us several checks on our curvature calculations. It is known that $(\mathbb{C}P^3, \gamma_{FS})$ has constant holomorphic sectional curvature (i.e. $\mathrm{Hol}(X)$ is independent of both $X \in T_p\mathbb{C}P^3$ and base point p), and is Einstein [39]. So substituting (4.21) into (4.4), (4.6) and (4.16) should yield constants. This is easily checked. One finds,

$$\mathrm{Hol}_{FS}(e_1) \equiv \mathrm{Hol}_{FS}(e_3) \equiv 4, \qquad \kappa_{FS} \equiv 48. \tag{4.22}$$

Also, substituting (4.21) into (4.10) demonstrates that $\bar{A}_{FS} = 8A_{FS}$, as it should. This gives us considerable confidence in the somewhat complicated expressions for Hol, ρ and κ.

5. Hamiltonian flows

The Kähler form Ω is a closed 2-form, nondegenerate by nondegeneracy of γ, and hence a natural symplectic form on M_1. Associated with any smooth function $H : \mathrm{M}_1 \to \mathbb{R}$ there is a Hamiltonian flow, defined as the flow along the smooth vector field X_H defined such that

$$\Omega(Y, X_H) = \mathrm{d}H(Y) \tag{5.1}$$

for all vector fields Y. Thinking of M_1 as the 1-lump moduli space, only $SO(3) \times SO(3)$ invariant Hamiltonians make physical sense, so H must be a function of λ only.

Proposition 5.1. *Let Ω be the Kähler form associated with a G invariant Kähler metric on M_1, determined as in Proposition 3.1 by $A(\lambda)$, and $H(\lambda)$ be a smooth, G invariant function on M_1. The Hamiltonian vector field corresponding to (Ω, H) is*

$$X_H = \frac{2\sqrt{1+\lambda^2}H'(\lambda)}{(1+2\lambda^2)A(\lambda) + (\lambda+\lambda^3)A'(\lambda)}\hat{\boldsymbol{\lambda}} \cdot \boldsymbol{\theta}. \tag{5.2}$$

Proof. It is convenient to decompose vector fields relative to the moving frame $\{\partial/\partial\lambda_1, \dots, \theta_3\}$ using the notation

$$Y = \mathbf{Y} \cdot \frac{\partial}{\partial \boldsymbol{\lambda}} + \tilde{\mathbf{Y}} \cdot \boldsymbol{\theta}, \tag{5.3}$$

that is, collecting the coefficients into a pair of $\mathbb{R}^3$-vector valued functions. Recall from the proof of Proposition 3.1 that the Kähler form is

$$\Omega = \hat{A}_1(\mathrm{d}\boldsymbol{\lambda} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathrm{d}\boldsymbol{\lambda}) + \hat{A}_2(\boldsymbol{\lambda} \cdot \mathrm{d}\boldsymbol{\lambda}) \wedge (\boldsymbol{\lambda} \cdot \boldsymbol{\sigma}) + \hat{A}_1\boldsymbol{\lambda} \cdot (\boldsymbol{\sigma} \times \boldsymbol{\sigma}), \tag{5.4}$$

so the defining equation for the Hamiltonian vector field $X_H = \mathbf{X} \cdot \partial/\partial\boldsymbol{\lambda} + \tilde{\mathbf{X}} \cdot \boldsymbol{\theta}$ reads

$$\begin{aligned} &\hat{A}_1(\mathbf{Y} \cdot \tilde{\mathbf{X}} - \tilde{\mathbf{Y}} \cdot \mathbf{X}) + \hat{A}_2[(\boldsymbol{\lambda} \cdot \mathbf{Y})(\boldsymbol{\lambda} \cdot \tilde{\mathbf{X}}) - (\boldsymbol{\lambda} \cdot \tilde{\mathbf{Y}})(\boldsymbol{\lambda} \cdot \mathbf{X})] + \hat{A}_1\boldsymbol{\lambda} \cdot (\tilde{\mathbf{Y}} \times \tilde{\mathbf{X}}) \\ &\quad = \frac{H'}{\lambda}\boldsymbol{\lambda} \cdot \mathbf{Y} \quad \forall Y \Rightarrow \hat{A}_1\mathbf{X} + \hat{A}_2(\boldsymbol{\lambda} \cdot \mathbf{X})\boldsymbol{\lambda} + \hat{A}_1\boldsymbol{\lambda} \times \tilde{\mathbf{X}} = \mathbf{0}, \end{aligned} \tag{5.5}$$

$$\hat{A}_1\tilde{\mathbf{X}} + \hat{A}_2(\boldsymbol{\lambda} \cdot \tilde{\mathbf{X}})\boldsymbol{\lambda} - \frac{H'}{\lambda}\boldsymbol{\lambda} = \mathbf{0}. \tag{5.6}$$

The pair (5.5) and (5.6) is easily solved for $\mathbf{X}$, $\tilde{\mathbf{X}}$, yielding

$$X_H = \frac{H'}{\hat{A}_1 + \lambda^2 \hat{A}_2} \hat{\boldsymbol{\lambda}} \cdot \boldsymbol{\theta}. \tag{5.7}$$

One now uses (3.7) and (3.27) to rewrite $\hat{A}_1$ and $\hat{A}_2$ in terms of A. □

Flow along X_H corresponds physically to a lump which maintains constant shape λ and position $-\hat{\boldsymbol{\lambda}}$, while spinning internally at constant speed about its axis. The variation of spin speed and sense with λ depends on the specifics of $H(\lambda)$.

6. The space of harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$

We begin by recalling some relevant results of Eells and Lemaire [6]. The homotopy classes of continuous maps $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ fall into distinct families labelled by the induced endomorphism of the fundamental group, $\phi_* : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$. Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, there are two families, one for which ϕ_* is the zero morphism (ϕ maps all loops to contractible loops), the other where ϕ_* is the identity morphism (ϕ maps noncontractible loops to noncontractible loops). The zero morphism family contains two classes, one of which is the trivial class. The identity morphism family contains infinitely many classes. Any map in this family lifts to $\tilde{\phi} : S^2 \to S^2$,

$$\begin{array}{ccc} S^2 & \xrightarrow{\tilde{\phi}} & S^2 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{R}P^2 & \xrightarrow{\phi} & \mathbb{R}P^2 \end{array} \tag{6.1}$$

where π denotes the covering projection, and the different classes are distinguished by the absolute value of the degree of $\tilde{\phi}$, which may take any *odd* value. We shall refer to this homotopy invariant as the absolute degree $|\deg|$ of ϕ.

Turning to harmonic maps, all but one of the homotopy classes described above contain harmonic representatives. Again following [6], if ϕ belongs to the zero morphism family, it lifts to a map $\bar{\phi} : \mathbb{R}P^2 \to S^2$ which is also harmonic since the covering projection $\pi : S^2 \to \mathbb{R}P^2$ is a local isometry. All harmonic maps from $\mathbb{R}P^2$ to S^2 are constant, so the nontrivial class has no harmonic representative. The moduli space of harmonic maps in the trivial class is thus $\mathbb{R}P^2$, and the L^2 metric on this space is a constant multiple of the canonical metric. If ϕ is harmonic and belongs to the identity morphism family, it lifts to a harmonic map $\tilde{\phi} : S^2 \to S^2$ (again, because π is a local isometry), and the space of these is well understood in terms of rational maps. So the task is to identify those harmonic maps $\tilde{\phi} : S^2 \to S^2$ which factor through the quotient in (6.1). Let $p : S^2 \to S^2$ be the antipodal map ($p : z \mapsto -1/\bar{z}$ in stereographic coordinates). Then $\tilde{\phi}$ projects to a well-defined map $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ if and only if $\tilde{\phi} \circ p = p \circ \tilde{\phi}$, or in terms of the associated rational map $W(z)$,

$$[\overline{W(z)}]^{-1} = W(\bar{z}^{-1}). \tag{6.2}$$

We now note that given such a rational map, of degree $n > 0$ say, no other degree n map projects to the same ϕ, although $W(-1/\bar{z})$, which has degree $-n$, does. So we may identify $\tilde{\mathrm{M}}_n$, the moduli space of $|\deg|\ n$ harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$, with the subset of M_n on which (6.2) holds.

Theorem 6.1. $\tilde{\mathrm{M}}_n$, *where* $n \geq 1$ *is odd, is a totally geodesic Lagrangian submanifold of* $(\mathrm{M}_n, \gamma, \Omega)$.

Proof. Let $\mathsf{P} : \mathrm{M}_n \to \mathrm{M}_n$ such that

$$\mathsf{P} : \tilde{\phi} \mapsto p \circ \tilde{\phi} \circ p. \tag{6.3}$$

Then $\tilde{\mathrm{M}}_n \subset \mathrm{M}_n$ is precisely the fixed point set of P. Since P is an isometry of (M_n, γ), in the component $\overline{\mathrm{SO}(3)} \times \overline{\mathrm{SO}(3)}$, $\tilde{\mathrm{M}}_n$ is totally geodesic if it is a submanifold (i.e. nonsingular). Extending P naturally to $\mathbb{C}P^{2n+1}$, one finds that

$$\begin{aligned} \mathsf{P} : & [a_1, \ldots, a_{n+1}, a_{n+2}, \ldots, a_{2n+2}] \\ & \mapsto [(-1)^n \bar{a}_{2n+2}, (-1)^{n-1} \bar{a}_{2n+1}, \ldots, -\bar{a}_{n+3}, \bar{a}_{n+2}, (-1)^{n+1} \bar{a}_{n+1}, \\ & \quad (-1)^n \bar{a}_n, \ldots, \bar{a}_2, -\bar{a}_1], \end{aligned} \tag{6.4}$$

which is manifestly antiholomorphic. Hence $\mathsf{P}^* \Omega = -\Omega$, and the Kähler (symplectic) form restricts to 0 on the fixed point set. So $\tilde{\mathrm{M}}_n$ is a Lagrangian submanifold if it is nonsingular and has (real) dimension $2n + 1$.

It remains to check that $\tilde{\mathrm{M}}_n$ is indeed nonsingular and has half the dimension of M_n. A short calculation in inhomogeneous coordinates demonstrates that the fixed point set of P in $\mathbb{C}P^{2n+1}$ is smooth with real dimension $2n + 1$ if n is odd, and is empty if n is even (the latter being a special case of the topological fact that no even degree map $S^2 \to S^2$ projects to a map $\mathbb{R}P^2 \to \mathbb{R}P^2$ in (6.1)). This does not suffice for our purposes, however, since a real codimension 2 algebraic variety must be removed from $\mathbb{C}P^{2n+1}$ to yield M_n. We must verify, therefore, that the intersection of $\tilde{\mathrm{M}}_n$ with this singular set has dimension lower than $2n + 1$.

Since the question is local, we may work in a neighbourhood of any fixed map $\tilde{\phi}$, and choose stereographic coordinates on the codomain which are projected from neither $\tilde{\phi}((0, 0, 1))$ nor $\tilde{\phi}((0, 0, -1))$. Then, in a sufficiently small neighbourhood, all harmonic maps have rational form

$$W(z) = \mu \frac{(z - z_1) \cdots (z - z_n)}{(z - w_1) \cdots (z - w_n)}, \tag{6.5}$$

where $\mu \in \mathbb{C}^\times$. These should be thought of as parameterized by μ and a pair of *unordered* n-tuples of complex numbers $\{w_i\}$, $\{z_i\} \in \mathbb{C}^n / P_n$, P_n being the permutation group on n objects. Of course, in this context $\mathbb{C}^n / P_m \cong \mathbb{C}^n$ diffeomorphically through the global coordinates $\{a_i\}$, where $(z - z_1) \cdots (z - z_n) =: z^n + a_n z^{n-1} + \cdots + a_1$. The singular set, on which $\deg W < n$, is that piece where $\{w_i\} \cap \{z_i\} \neq \emptyset$. The fixed point set of P in this neighbourhood consists of maps for which

$$\{z_1, \ldots, z_n\} = \left\{ -\frac{1}{\bar{w}_1}, \ldots, -\frac{1}{\bar{w}_n} \right\}, \tag{6.6}$$

and

$$|\mu| = |w_1 w_2 \cdots w_n|. \tag{6.7}$$

Eqs. (6.6) and (6.7) determine a $(2n+1)$-dimensional submanifold of $\mathbb{C}^\times \times [\mathbb{C}^n/P_n] \times [\mathbb{C}^n/P_n]$, parameterized by $\{w_i\} \in [\mathbb{C}^\times]^n/P_n$ and $\arg\mu \in S^1$. From this must be excluded, if $n \geq 3$, the $(2n-3)$-dimensional variety on which $w_i = -1/\bar{w}_j$ for some i, j. This still leaves a nonsingular $(2n+1)$-dimensional fixed point set, as was to be proved. $\square$

Note that P is also an antiholomorphic isometry of γ_{FS}, the Fubini–Study metric inherited from the open inclusion $\mathrm{M}_n \subset \mathbb{C}P^{2n+1}$. So $\tilde{\mathrm{M}}_n$ is a totally geodesic Lagrangian submanifold of $(\mathrm{M}_n, \gamma_{\mathrm{FS}}, \Omega_{\mathrm{FS}})$ also, by identical reasoning. The metric induced on $\tilde{\mathrm{M}}_n$ by γ is more interesting than that induced by γ_{FS}, however, since it coincides with the L^2 metric on $\tilde{\mathrm{M}}_n$. The geodesic approximation to $\mathbb{R}P^2$ lump dynamics on $\mathbb{R}P^2$ is thus a special case of S^2 lump dynamics on S^2.

It is clear from the proof above that $\tilde{\mathrm{M}}_n$ is generically noncompact. The case $n = 1$ is exceptional, however. Here, as described in Section 3, one may identify a rational map with a projective equivalence class $[M]$ of $\mathrm{GL}(2,\mathbb{C})$ matrices. Let $[M]$ be a fixed point of $\mathsf{P} : \mathrm{PL}(2,\mathbb{C}) \to \mathrm{PL}(2,\mathbb{C})$. Then

$$\mathsf{P} : \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} -\bar{a}_{22} & \bar{a}_{21} \\ \bar{a}_{12} & -\bar{a}_{11} \end{pmatrix}\right] = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]. \tag{6.8}$$

So there exists $\xi \in \mathbb{C}^\times$ such that

$$a_{11} = -\xi\bar{a}_{22}, \qquad a_{12} = \xi\bar{a}_{21}, \qquad a_{21} = \xi\bar{a}_{12}, \qquad a_{22} = -\xi\bar{a}_{11}, \tag{6.9}$$

whence it follows that $|\xi| = 1$. But then

$$MM^\dagger = \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 & a_{11}\bar{a}_{21} + a_{12}\bar{a}_{22} \\ a_{21}\bar{a}_{11} + a_{22}\bar{a}_{12} & |a_{21}|^2 + |a_{22}|^2 \end{pmatrix} = (|a_{11}|^2 + |a_{12}|^2)\mathbb{I}_2, \tag{6.10}$$

so $[M] \in \mathrm{PU}(2) \cong \mathrm{SO}(3)$. Hence $\tilde{\mathrm{M}}_1$ consists of the rotation orbit of Id: $z \mapsto z$, and the induced metric $\tilde{\gamma}$ on $\tilde{\mathrm{M}}_1$ is

$$\tilde{\gamma} = A_3(0)\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}, \tag{6.11}$$

the standard bi-invariant metric on SO(3), up to a constant factor. Each $\phi \in \tilde{\mathrm{M}}_1$ has completely uniform energy density, so it is rather misleading to call these solutions "$\mathbb{R}P^2$ lumps".

For higher n the possibilities are more varied. For example, the energy density of $[z \mapsto z^n] \in \tilde{\mathrm{M}}_n$, $n \geq 3$, is concentrated in a symmetric band centred on a (projected) great circle on $\mathbb{R}P^2$, the band being narrower for larger n. Considering rational maps of the form (6.5), with parameters satisfying (6.6) and (6.7), a sharp lump-like structure may be induced by arranging that one of the poles of W be close to one of the zeroes, for example by choosing w_2 close to $-1/\bar{w}_1$, while keeping the other poles and zeroes well separated. Since lumps are associated with close pole-zero *pairs*, and the poles determine the zeroes (they must be antipodal), for $\phi \in \tilde{\mathrm{M}}_n$ at most $(n-1)/2$ distinct lumps in the energy distribution are possible.

The origin of the noncompactness of $\tilde{\mathrm{M}}_n$, $n \geq 3$, is that when $w_2 \to -1/\bar{w}_1$, say, the degree of $\tilde{\phi}$ drops by 2, that is a lump (or, in the lifted picture, an antipodal pair of lumps) forms, collapses to an infinitely sharp spike and disappears. In fact, there are geodesics with respect to $\tilde{\gamma}$ which reach such singularities in finite time. We conclude this section by establishing the following theorem.

Theorem 6.2. *For all $n \geq 3$, $(\tilde{\mathrm{M}}_n, \tilde{\gamma})$ is geodesically incomplete.*

Proof. It suffices [17] to exhibit a curve of finite length which converges to infinity, that is, escapes every compact subset of $\tilde{\mathrm{M}}_n$. Consider the curve $\Gamma : [1/2, 1) \ni \rho \mapsto W_\rho \in \tilde{\mathrm{M}}_n$, where

$$W_\rho(z) = \rho z^{n-2} \frac{(z+1)(z-\rho^{-1})}{(z-1)(z+\rho)}, \tag{6.12}$$

which certainly converges to infinity (as $\rho \to 1$). The induced metric on Γ is $\tilde{\gamma}_\Gamma = f(\rho)\,\mathrm{d}\rho^2$, where

$$f(\rho) = \int_{\mathbb{C}} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2} \frac{1}{(1+|W_\rho|^2)^2} \left| \frac{\partial W_\rho}{\partial \rho} \right|^2 . \tag{6.13}$$

We now appeal to a technical lemma whose proof is postponed to Appendix A.

Lemma 6.3. *There exist $C > 0$ and $\rho_* \in (0, 1)$ such that for all $\rho \in (\rho_*, 1)$,*

$$f(\rho) < C \left[1 + \log\left(\frac{1}{1-\rho} \right) \right].$$

Hence, the length of Γ

$$\int_{1/2}^{1} \mathrm{d}\rho \sqrt{f(\rho)} < C \left[1 + \int_{\rho_*}^{1} \mathrm{d}\rho \sqrt{1 + \log\left(\frac{1}{1-\rho} \right)} \right] \tag{6.14}$$

is finite. □

Note that this result does not follow directly from the results of Sadun and Speight [29] previously mentioned (incompleteness of M_n), although the method of proof is similar. Recall that geodesic flow on $(\tilde{\mathrm{M}}_n, \tilde{\gamma})$ is conjectured to approximate closely the low energy dynamics of the $\mathbb{R}P^2$ σ-model on space–time $\mathbb{R}P^2 \times \mathbb{R}$. So the geodesic approximation predicts that $\mathbb{R}P^2$ lumps on $\mathbb{R}P^2$ may collapse and form singularities in finite time, just as it does for S^2 lumps on any compact Riemann surface. In fact, little is known about singularity formation in the full $(2+1)$-dimensional system, although there is some numerical evidence in favour of lump collapse [21,26].

7. Concluding remarks

One could hope to generalize the results of this paper in at least two directions. Replacing the domain 2-sphere by an arbitrary compact Riemann surface Σ, one could study the L^2 metric on the space $\mathrm{Hol}_n(\Sigma)$ of degree n (anti)holomorphic maps $\Sigma \to S^2$. If nonempty, $\mathrm{Hol}_n(\Sigma)$ is the space of *minimal energy* degree n harmonic maps (if empty, for example $\mathrm{Hol}_{\pm 1}(T^2) = \emptyset$, there exists no minimal energy degree n harmonic map), which is the space of most direct interest to physicists, rather than the space of *all* harmonic maps. $\mathrm{Hol}_n(\Sigma)$ has the structure of a complex algebraic variety, so one would expect Theorem 2.1, the Kähler property of the L^2 metric, to generalize to this situation. (In fact, $\mathrm{Hol}_n(\Sigma)$ may not be smooth if $|n| \leq 2\,\mathrm{genus}(\Sigma) - 2$, but the Kähler property should still hold in the complement of the singular set.)

As an example, consider $\mathrm{Hol}_2(T^2)$. It was proved in [33] that $\mathrm{Hol}_2(T^2)$ is homeomorphic (in C^0 topology) to the complex homogeneous space $[\mathrm{PL}(2,\mathbb{C}) \times T^2]/V_4$, where V_4 is a certain Viergruppe (finite group of order 4, each element being its own inverse). So $\mathrm{Hol}_2(T^2)$ inherits a natural complex structure from the covering space $\mathrm{PL}(2,\mathbb{C}) \times T^2$, and it suffices to show that the lift of the L^2 metric is Kähler. Explicitly, a point

$$\left(\left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right], s\right) \in \mathrm{PL}(2,\mathbb{C}) \times T^2, \tag{7.1}$$

is identified with the degree 2 holomorphic map

$$W(z) = \frac{a_1 \wp(z-s) + a_2}{a_3 \wp(z-s) + a_4}, \tag{7.2}$$

where $\wp$ is the Weierstrass p-function. Introducing inhomogeneous coordinates on $\mathrm{PL}(2,\mathbb{C})$, an essentially identical argument to that of the proof of Theorem 2.1 establishes.

Theorem 7.1. *The L^2 metric γ on $\mathrm{Hol}_2(T^2)$ is Kähler with respect to the complex structure induced by the identification with* $[\mathrm{PL}(2,\mathbb{C}) \times T^2]/V_4$.

It is interesting to note that $(\mathrm{Hol}_2(T^2), \gamma)$, like (M_1, γ) has finite diameter, leading one to expect that Theorem 3.3 should generalize to $(\mathrm{Hol}_n(\Sigma), \gamma)$ also.

The second natural generalization would be to replace the codomain $S^2 \cong \mathbb{C}P^1$ by a general projective space, $\mathbb{C}P^N$. Lemaire and Wood [19] have shown that the space of degree n, energy $4\pi E$ harmonic maps $S^2 \to \mathbb{C}P^2$, $\mathrm{Harm}_{n,E}(\mathbb{C}P^2)$ is, in C^j topology ($j \geq 2$), a disjoint union of smooth manifolds indexed by total ramification index. Further, there is an explicit identification between each smooth component of $\mathrm{Harm}_{n,E}(\mathbb{C}P^2)$ and an appropriate space of linearly full holomorphic maps $S^2 \to \mathbb{C}P^2$ of fixed degree and ramification index. So again one has a natural complex structure on the moduli space, and again one would expect the L^2 metric to be Kähler with respect to this structure. It is even possible that the Kähler property of the L^2 metric may persist when the codomain itself is *not* Kähler. Bolton and Woodward [3] have conjectured that $\mathrm{Harm}_E(S^{2m})$, the space of energy $4\pi E$ harmonic maps $S^2 \to S^{2m}$, is a complex algebraic variety (of dimension $2E+m^2$). If true, it would be natural to ask again whether γ is Kähler, at least on the smooth part of $\mathrm{Harm}_E(S^{2m})$. Note that both these generalizations lie beyond the scope of Ruback's formal argument [28].

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Appendix A. Proofs of Lemmas 2.2 and 6.3

Proof of Lemma 2.2. Since $F : X \times (-\epsilon, \epsilon) \to \mathbb{R}$ is smooth, its partial derivative with respect to the second entry, F_2 is continuous. Hence the restriction $\tilde{F}_2 : X \times [0, x] \to \mathbb{R}$, $0 < x < \epsilon$, is integrable (its domain is compact). Thus, by the Fubini theorem [4]

$$\int_X \left\{ \int_{[0,x]} \tilde{F}_2 \right\} = \int_{[0,x]} \left\{ \int_X \tilde{F}_2 \right\}.$$

But $\int_{[0,x]} \tilde{F}_2 \equiv F(\cdot, x) - F(\cdot, 0)$, so the left-hand side is $f(x) - f(0)$. Hence, by the fundamental theorem of calculus

$$f'(0) = \left\{ \int_X \tilde{F}_2 \right\}\bigg|_{x=0} = \int_X F_2(\cdot, 0). \qquad \square$$

Proof of Lemma 6.3. From (6.12) and (6.13) one finds that

$$f(\rho) = \int_{\mathbb{C}} \mathrm{d}z\, \mathrm{d}\bar{z}\, F(z, \rho),$$

where

$$F(z, \rho) = \frac{|1 + z^2|^2}{(1 + |z|^2)^2} \frac{|z|^{2(n-2)}|z + 1|^2|z - 1|^2}{(|z + \rho|^2|z - 1|^2 + |\rho z - 1|^2|z + 1|^2|z|^{2(n-2)})^2}.$$

Fix $\epsilon \in (0, 1/2)$, and assume that ρ is close to 1, that is $0 < \rho - 1 < \epsilon$. Then $F(\cdot, \rho)$ may be bounded independent of ρ except on the union of disks $D_\epsilon(-1) \cup D_\epsilon(1)$, where one or other of the terms in the denominator may vanish (here $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$). We shall denote positive constants (independent of z and ρ) by C_1, C_2, etc. On $D_\epsilon(-1)$ there exists C_1 such that

$$F(z, \rho) < \frac{C_1|z + 1|^2}{(|z + \rho|^2 + \alpha\rho^2|z + 1|^2)^2},$$

where $\alpha = (1-\epsilon)^{2(n-2)} < 1$. Hence, defining $r\,e^{i\theta} := z+1$,

$$\int_{D_\epsilon(-1)} dz\, d\bar{z}\, F(z,\rho)$$

$$< C_1 \int_0^{2\pi} d\theta \int_0^{\epsilon} dr \frac{r^3}{[(1+\alpha\rho^2)r^2 - 2(1-\rho)\cos\theta r + (1-\rho)^2]^2}$$

$$< C_2 \int_0^1 dr\, \frac{r^3}{[(3/2)r^2 - 2(1-\rho)r + (1-\rho)^2]^2},$$

provided $\rho > (2\alpha)^{-1/2} \in (0,1)$. Then, rescaling $r \mapsto r/(1-\rho)$, one finds that

$$\int_{D_\epsilon(-1)} dz\, d\bar{z}\, F(z,\rho) < C_2 \int_0^{(1-\rho)^{-1}} dr\, \frac{r^3}{[(3/2)r^2 - 2r^2 + 1]^2}$$

$$< C_3 + C_4 \int_1^{(1-\rho)^{-1}} \frac{dr}{r} < C_5[1 - \log(1-\rho)].$$

Noting that $\rho > (2\alpha)^{-1/2}$ implies $1+\alpha\rho^2 > 3\alpha/2$, one finds a similar estimate for the contribution from $D_\epsilon(1)$:

$$\int_{D_\epsilon(1)} dz\, d\bar{z}\, F(z,\rho) < C_6 \int_{D_\epsilon(1)} dz\, d\bar{z}\, \frac{|z-1|^2}{(|z-1|^2 + \alpha|\rho z - 1|^2)^2}$$

$$< C_7 \int_0^1 dr\, \frac{r^3}{[(1+\alpha\rho^2)r^2 - 2\alpha\rho(1-\rho)r + \alpha(1-\rho)^2]^2}$$

$$< C_8 \int_0^{(1-\rho)^{-1}} dr\, \frac{r^3}{[(3/2)r^2 - 2r + 1]^2} < C_9[1 - \log(1-\rho)].$$

Since F is bounded independent of ρ on $U = \mathbb{C}\backslash[D_\epsilon(-1) \cup D_\epsilon(1)]$,

$$\int_U dz\, d\bar{z}\, F(z,\rho) < C_{10} + C_{11} \int_1^{\infty} \frac{dr}{r^{2n-1}} < C_{12}.$$

Defining $C = C_5 + C_9 + C_{12} > 0$ and $\rho_* = (2\alpha)^{-1/2} \in (0,1)$, the lemma is proved. □

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